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Stone-Weierstrass Theorems for Function Spaces*

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1. INTRODUCTION

Let (Ω, Σ, μ) be a measure space such that each point of Ω is measurable and let $L^\rho(\Omega, \Sigma, \mu)$, or L^ρ for short, be the subset of the class of all (equivalence classes of) measurable scalar functions f on (Ω, Σ, μ) such that $\rho(f) < \infty$, where $\rho(\cdot)$ is a function norm, i.e., (i) $\rho(f) = \rho(|f|) \geqslant 0$ and $=0$ if and only if $f = 0$, a.e., (ii) $\rho(f_1 + f_2) \leqslant \rho(f_1) + \rho(f_2)$, (iii) f_i real and $f_1 \leqslant f_2$ a.e. implies $\rho(f_1) \leqslant \rho(f_2)$, (iv) $\rho(kf) \leqslant k\rho(f)$, $k \geqslant 0$, and (v) f_n real, $f_n \nearrow f$ a.e., and $\rho(f_n) \leqslant k_0 < \infty$ implies $\rho(f) < \infty$. Under these conditions L^ρ is a complete normed linear space and is called a (Banach) *function space*. (See [1] where L^ρ is denoted by L^λ, and λ called a length function, and [2]. The notation of the latter paper will be followed here. Luxemburg and Zaanen have studied the L^ρ spaces extensively and a reasonably complete set of references to the work of these authors and that of the subject may be found in [3].) If L^ρ is the familiar Lebesgue space L^p, $1 \leqslant p < \infty$, then Farrell [4] has proved an approximation theorem, and some of its consequences. The result may be considered as a "measure theoretic analog" of the Stone-Weierstrass Theorem for function algebras. [Here and below $f \in L^\rho$ means f is any member of its equivalence class.] See also Rota's paper [12].

The purpose of this paper is to prove a general approximation theorem for the spaces L^ρ, which subsumes the results of [4]. This result also includes the Orlicz spaces as special cases, and the treatment clarifies the algebraic structure of the problem. The (purely measure theoretic) argument given in [4] for the special case of L^p spaces, does not generalize to the present case because bounded functions are not, in general, dense in the L^ρ spaces, or even in the Orlicz spaces. This makes even the formulation of the results somewhat different. After giving alternate versions and specializations, a general form of a problem in probability and statistics will be presented, as an application. It includes the corresponding result given in [4], and in both cases this application is one of the motivations for the study of such problems.

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Thus Theorems 2.1 and 3.4 are the main results of this note and the others may be considered useful supplementaries to these results.

It must be noted that an L^p-space result related to that mentioned above appears in Rota's paper [12], earlier than [4], and was used in the main theory of [12].

2. Approximation Theory

A measurable function f is said to be *ρ-bounded* if $\rho(f) < \infty$. One form of the Stone-Weierstrass type approximation in the context of function spaces, can be formulated as follows.

Theorem 2.1. *Let $L^\rho(\Omega, \Sigma, \mu)$, or $L^\rho(\Sigma)$, be the space of scalar (measurable) functions on a complete measure space (Ω, Σ, μ) in which points of Ω are measurable. Suppose $\mathscr{C}_0 \subset L^\rho$ be an algebra of essentially bounded functions satisfying the following conditions:*

(1) *$\mathscr{C}_0$ is self-adjoint, i.e., $f \in \mathscr{C}_0$ implies its conjugate $f^* \in \mathscr{C}_0$,*

(2) *there is an $f_0 \in \mathscr{C}_0$ such that $f_0 > 0$, a.e., $[\mu]$,*

(3) *for every pair of disjoint measurable sets E_1, E_2 of Σ, there is a $g \in \mathscr{C}_0$ such that $g > 0$, a.e., on E_1 and $g \leqslant 0$, a.e., on E_2.*

If the σ-field generated by $\mathscr{C}_0$, is denoted by Σ_0 and $L^\rho(\Sigma_0)$ is the corresponding subspace of $L^\rho(\Sigma)$ of functions measurable relative to Σ_0, then $L^\rho(\Sigma_0) = L^\rho(\Sigma)$. Equivalently, the class of all ρ-bounded scalar functions which are Σ_0-measurable is $L^\rho(\Sigma)$ itself.

A slightly stronger, and more appealing, conclusion can be given if $\rho(\cdot)$ has the following somewhat stronger property than (v) above: (v′) if $0 \leqslant f_n \nearrow f$, a.e., and f_n are Σ-measurable, then $\rho(f_n) \nearrow \rho(f)$. The property (v′) was called the *strong Fatou property* and (v) itself was termed the *weak Fatou property* in [2]. An element $f \in L^\rho$ is said to have an *absolutely continuous* norm if $\rho(f\chi_{E_n}) \searrow 0$ (χ_A is the characteristic function of A) for every measurable $E_n \searrow \phi$. Then one has:

Theorem 2.2. *If L^ρ is the function space on (Ω, Σ, μ) with ρ having the strong Fatou property, let $\mathscr{C}_0 \subset L^\rho$ be the algebra of essentially bounded functions satisfying conditions (1)–(3) of Theorem 2.1. Then for every $f \in L^\rho$, there exists a sequence $\{f_n\}$ of essentially bounded Σ_0-measurable functions, such that $f_n \to f$ a.e., and $\lim_{n\to\infty} \rho(f_n) = \rho(f)$. Moreover, if every element of L^ρ has an absolutely continuous norm, then $\mathscr{C}_0$ is (norm) dense in L^ρ. Here again Σ_0 is the σ-field generated by $\mathscr{C}_0$.*

It will be seen in the next section that the formulation of Theorem 2.2 is useful for specializations and applications, and in particular the main result of [4] is subsumed under its last part. Two proofs of Theorem 2.1 will be presented below as they have some independent interest and may be useful in further generalizations.

FIRST PROOF OF THEOREM 2.1. Let $\mathscr{C}_1 \subset L^p(\Sigma)$ be the algebra of all (essentially) bounded functions and suppose Σ_1 is the σ-field determined by $\mathscr{C}_1$. Then $\Sigma_1 \subset \Sigma$ and if $L^p(\Sigma_1)$ is the set of all Σ_1-measurable functions of $L^p(\Sigma)$, then $L^p(\Sigma_1)$ and $L^p(\Sigma)$ determine the same B-space when functions differing on null sets are identified, i.e., $L^p(\Sigma_1) = L^p(\Sigma)$. For, since $L^p(\Sigma_1) \subset L^p(\Sigma)$ is always true, let $f \in L^p(\Sigma)$. If $f_n = f$ on $[|f| \leqslant n]$ and $= n$ on its complement, then $f_n \in \mathscr{C}_1 \subset L^p(\Sigma)$ for each n and $f_n \to f$, a.e. So f_n is Σ_1-measurable and hence also is f. But $f_n \in L^p(\Sigma_1)$ and, by the weak Fatou property, $f \in L^p(\Sigma_1)$. Hence $L^p(\Sigma_1) \supset L^p(\Sigma)$ so that $L^p(\Sigma_1) = L^p(\Sigma)$.

Let $\mathscr{C}_0 \subset \mathscr{C}_1$ as in the theorem and suppose Σ_0 is the σ-field determined by $\mathscr{C}_0$. To prove $L^p(\Sigma_0) = L^p(\Sigma)$, and hence the theorem, it suffices to show $\Sigma_1 = \Sigma_0$ or equivalently it suffices to show that $\bar{\mathscr{C}}_0 = \bar{\mathscr{C}}_1$ where the "bar" denotes the completion of these spaces under the essential supremum norm (i.e., subspaces of $L^\infty(\mu)$).

First suppose $L^p(\Sigma_1)$ is the real function space. The complex case can then be deduced from the real case. Now $\mathscr{C}_0$ and $\mathscr{C}_1$, are abstract (M)-spaces and by a representation theorem of Kakutani for such spaces (cf. [5]), each of these spaces is isometric and lattice isomorphic to a subspace of the space of all real continuous functions $C(S)$ on some compact Hausdorff space S. Since $\mathscr{C}_0$ and hence $\mathscr{C}_1$ satisfy conditions (2) and (3) of the theorem, the corresponding subspaces, say C_0 and C_1 of $C(S)$ possess the properties: (a) there is an $\hat{f}_0 > 0$ on S such that $\hat{f}_0 \in C_0 \subset C_1$, and ($b$) C_0, C_1 separate points of S, i.e., $s_1, s_2 \in S$ implies there is $\hat{f} \in C_0$ such that $\hat{f}(s_1) > 0, \hat{f}(s_2) \leqslant 0$. But then by the Stone-Weierstrass Theorem, in the form given in ([6], p. 9), either $\bar{C}_0 = C(S)$ or all the functions of $\bar{C}_0$ vanish at a single point of S, where $\bar{C}_i$ is the uniform closure of the algebra C_i, $i = 0, 1$. Condition (a) prevents the second possibility. Hence $\bar{C}_0 = \bar{C}_1 = C(S)$. By the isometry and isomorphism, it follows that $\bar{\mathscr{C}}_0 = \bar{\mathscr{C}}_1$ and hence $\Sigma_0 = \Sigma_1$. This and the first paragraph above imply $L^p(\Sigma_0) = L^p(\Sigma)$.

In the complex case, the real functions of $\mathscr{C}_0$ satisfy conditions (2) and (3) and constitute a subalgebra $\mathscr{C}_0{}^R$ of the real space $L_R{}^p(\Sigma)$. But $f \in L^p(\Sigma)$ implies $f = f_1 + if_2$, $f_i \in L_R{}^p(\Sigma)$, $i = 1, 2$, and by the preceding paragraph $f_i \in L_R{}^p(\Sigma_0)$, Σ_0 being the σ-field generated by $\mathscr{C}_0{}^R$. By condition (1) every element of $\mathscr{C}_0$ is a linear combination of elements of $\mathscr{C}_0{}^R$. From this it follows easily that $L^p(\Sigma_0) = L^p(\Sigma)$. This completes the proof.

SECOND PROOF. The argument now uses some elementary results from the Gel'fand represenation theory of Banach algebras (instead of the Stone-

Weierstrass theorem above) and a sketch of proof is as follows. Let $\bar{\mathscr{C}}_0$ and $\bar{\mathscr{C}}_1$ be as in the above proof. These are commutative, self-adjoint, separating Banach algebras and let Δ_0 and Δ_1 be the set of all continuous homomorphisms of $\bar{\mathscr{C}}_0$ and $\bar{\mathscr{C}}_1$. Then by the basic Gel'fand theory Δ_0 and Δ_1 are locally compact and since $f_0 \in \mathscr{C}_0 \subset \mathscr{C}_1, f_0 > 0$, a.e., it follows by ([16], Theorem 19B) that Δ_0 and Δ_1 are in fact compact. The elements of $\bar{\mathscr{C}}_0$ and $\bar{\mathscr{C}}_1$ can be considered (for this proof) as functions using the lifting property (cf. [7]). Moreover $\bar{\mathscr{C}}_0$ and $\bar{\mathscr{C}}_1$ are isometrically isomorphic to the spaces of all continuous complex functions $C(\Delta_0)$ and $C(\Delta_1)$ so that $\bar{\mathscr{C}}_0$ and $\bar{\mathscr{C}}_1$ are self-adjoint, separating and inverse-closed (in the sense of [6], p. 54). With this and ([6], Corollary on p. 55) one can identify Ω as a dense subset of Δ_0 and Δ_1 where the points of Ω are first identified with maximal ideals and then the latter with the Δ's thereafter. (See also [8], p. 78.) It follows that $\Delta_0 = \Delta_1(=\Omega)$, and hence $\bar{\mathscr{C}}_0 = \bar{\mathscr{C}}_1$. Consequently $L^\rho(\Sigma) = L^\rho(\Sigma_0)$ as before.

PROOF OF THEOREM 2.2. By the above theorem $L^\rho(\Sigma) = L^\rho(\Sigma_0) = L^\rho(\Sigma_1)$. For this proof it suffices to consider non-negative functions. If $0 \leqslant f \in L^\rho(\Sigma)$, and $f_n = \inf(f, n) \in L^\rho(\Sigma_1) = L^\rho(\Sigma_0)$, then $f_n \nearrow f$ a.e., and by the strong Fatou property of $\rho(\cdot)$, $\rho(f_n) \nearrow \rho(f) < \infty$. In other words f is the a.e. limit of the sequence f_n, of Σ_0-measurable (essentially) bounded functions with $\lim_{n\to\infty} \rho(f_n) = \rho(f)$. Finally, if every element of $L^\rho(\Sigma)$ has an absolutely continuous norm, and $0 \leqslant f \in L^\rho(\Sigma)$ so that there is a bounded sequence f_n such that $0 \leqslant f_n \nearrow f$, a.e., and $\rho(f_n) \nearrow \rho(f) < \infty$, as above, then by ([2], Theorem 2.2 which holds for arbitrary measures) it follows that $\rho(f - f_n) \to 0$. But since $f_n \in \mathscr{C}_0$, the algebra of all bounded functions in $L^\rho(\Sigma_0) = L^\rho(\Sigma)$, it follows that every f in $L^\rho(\Sigma_0) = L^\rho(\Sigma)$ can be approximated in norm by elements of $\mathscr{C}_0$, i.e., the norm closure of $\mathscr{C}_0$ in $L^\rho(\Sigma)$ is $L^\rho(\Sigma)$ itself. This completes the proof.

As special cases of the above, some results will be given for Orlicz spaces L^Φ, since they are of interest in applications. Briefly, L^Φ is the (sub-) space of the class of all measurable scalar functions f on (Ω, Σ, μ) for which $N_\Phi(f) < \infty$ where

$$N_\Phi(f) = \inf \left\{ k > 0 : \int_\Omega \Phi\left(\frac{|f|}{k}\right) d\mu \leqslant 1 \right\}, \tag{1}$$

and where Φ is a symmetric convex function on the line with $\Phi(0) = 0$, called a Young's function. It is known that $N_\Phi(\cdot)$ is a norm under which L^Φ is a Banach space (cf. [9] and the references there; and for a more comprehensive account of the general theory, see [10]).

COROLLARY 2.3. *Let Φ be a continuous Young's function with $\Phi(x) > 0$ for $x > 0$. If $\mathscr{C}_0 \subset L^\Phi$ is an algebra of essentially bounded functions satisfying* *conditions* (1) *to* (3) *of Theorem* 2.1, *then* $\mathscr{C}_0$ *is* (*norm*) *dense in* L^Φ *whenever* $\Phi(2x) \leqslant C\Phi(x)$, $x \geqslant 0$ *where* $0 < C < \infty$. *Without this last condition, it is always true that, for each* $f \in L^\Phi$, *there exists a sequence* $\{f_n\}$ *of bounded* Σ_0-*measurable functions in* L^Φ *such that* $f_n \to f$, *a.e., and* $N_\Phi(f_n) \to N_\Phi(f)$, *as* $n \to \infty$, *where* Σ_0 *is the* σ-*field determined by* $\mathscr{C}_0$.

PROOF. If Φ satisfies the above growth condition, then (a special case of Lemma 2 of [9] implies) every f in L^Φ has an absolutely continuous norm so that $\mathscr{C}_0$ is dense in L^Φ by Theorem 2.2. The last conclusion is a consequence of the fact that the norm $N_\Phi(\cdot)$ has the strong Fatou property and Theorem 2.2 is again applicable.

COROLLARY 2.4. *If* L^p, $1 \leqslant p < \infty$ *is a Lebesgue space of scalar functions on* (Ω, Σ, μ) *and* $\mathscr{C}_0 \subset L^p$ *is an algebra of essentially bounded functions which satisfies conditions* (1)–(3) *of Theorem* 2.1, *then* $\mathscr{C}_0$ *is dense in* L^p.

This result follows from the preceding corollary since $\Phi(x) = | x |^p$, and the growth condition is satisfied here.

REMARK. The result of this corollary is due to Farrell [4], who assumed that (Ω, Σ, μ) is a σ-finite regular Baire measure space with Ω as a locally compact set, and whose proof is different from the one given here. The conditions (1)–(3) are essentially the same as in [4], given for the L^p-case, but their algebraic significance was not fully apparant from the proof given there. The above proofs show (i) that these conditions cannot be materially weakened for this type of approximation theorems for function spaces, and (ii) that the topological nature of (Ω, Σ, μ) is unnecessary. Note that if ρ is replaced by a supremum norm, Σ the power set of Ω, and μ the point measure, then Theorem 2.1 is essentially the Stone-Weierstrass theorem (in the sense of [6], Theorem 26E). The above Cor. 2.4 was also given by Rota, [12].

3. AN APPLICATION

The application considered below is related to the problem of bounded completeness of probability measures on a measurable space (Ω, Σ). A family $\mathscr{F}$ of probability measures on Σ is *boundedly complete if* $\int_\Omega f\, d\mu = 0$, $\mu \in \mathscr{F}$, and f is bounded implies $f = 0$ a.e., $[\mathscr{F}]$. (Cf., e.g., [11], p. 134, for a discussion of its importance.) By assuming a slightly stronger condition a stronger conclusion will be obtained below even if the measures are not finite, and thus the application is not restricted to the above case. (Without some strengthening, no stronger conclusion is generally possible as easy counterexamples show. Cf., e.g., [11], p. 152, Ex. 11.)

Let L^ρ be a function space, as before, and let ρ' be the *associate norm* (cf. [2], p. 153 and [1], p. 243) defined by

$$\rho'(f) = \sup \left\{ \left| \int_\Omega fg \, d\mu \right| : \rho(g) \leqslant 1 \right\}. \tag{2}$$

An additive (scalar valued) set function G on Σ, vanishing on μ-null sets is said to be of *ρ-bounded variation*, relative to μ, if

$$V_\rho(G) = \sup_\pi \rho \left(\sum_{i=1}^n \frac{G(A_i)}{\mu(A_i)} \chi_{A_i} \right) < \infty, \tag{3}$$

where $\pi = \{A_i\} \subset \Sigma$ is a finite disjoint collection of sets with $0 < \mu(A_i) < \infty$, and the supremum is taken on all such $\pi \subset \Sigma$. This concept was introduced in [3], generalizing the *Φ-bounded* variation of ([9], p. 83), i.e.,

$$\sup_\pi \sum_{i=1}^n \Phi \left(\frac{G(A_i)}{\mu(A_i)} \right) \mu(A_i) = I_\Phi(G) < \infty.$$

The following technical lemma is useful below, and also has some independent interest.

LEMMA 3.1. *Let ρ be a function norm and ρ' its associate norm. If G on (Ω, Σ, μ) is an additive set function of ρ-bounded variation relative to μ, and ρ' is continuous* (i.e., *for any $\epsilon > 0$, there is a $\delta_\epsilon > 0$ such that for every $E \in \Sigma$ with $\mu(E) < \delta_\epsilon$, one has $\rho'(\chi_E) < \epsilon$*), *then G is μ-continuous, so that* $\lim_{\mu(A)\to 0} G(A) = 0$.

PROOF. Excluding the true and trivial case $G = 0$ (i.e., $V_\rho(G) = 0$), given $\epsilon > 0$, choose $A \in \Sigma$ such that $\mu(A) < \delta_\epsilon$ implies $\rho'(\chi_A) < \epsilon / V_\rho(G)$. If $\{A_i\}$ is a measurable dissection of A (i.e., $A_i \in \Sigma$, $\mu(A_i) > 0$, $A = \bigcup_{i=1}^n A_i$ and A_i disjoint), let $x_i = G(A_i)/\mu(A_i)$, and $x(\cdot) = \sum_{i=1}^n x_i \chi_{A_i}$. Then

$$\begin{aligned} | G(A)| &\leqslant \sum_{i=1}^n | x_i | \mu(A_i) = \int_\Omega \chi_A | x | \, d\mu \\ &\leqslant \rho(x) \cdot \rho'(\chi_A), \qquad \text{by} \quad ([2], \text{p. } 153). \\ &\leqslant V_\rho(G) \cdot \rho'(\chi_A) = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the lemma follows.

The following two consequences of this lemma are noteworthy. If $\Phi(\cdot)$ is a Young's function, let $\Psi(\cdot)$ be its complementary function, defined by $\Psi(x) = \sup_{y>0} \{| x | y - \Phi(y)\}$. Then $\Psi(\cdot)$ is also a Young's function. If μ

has the finite subset property (FSP), [i.e., if $A \in \Sigma$, $\mu(A) > 0$ implies the existence of $E \in \Sigma$, $E \subset A$ and $0 < \mu(E) < \infty$] then it is known that, with $\rho(\cdot) = N_\Phi(\cdot)$, the norms $N_\Psi(\cdot)$ and $\rho'(\cdot)$ are equivalent. In fact $N_\Psi(\cdot) \leqslant \rho'(\cdot) \leqslant 2N_\Psi(\cdot)$ holds true.

COROLLARY 3.2 ([9], Lemma 6). *If Φ is a Young's function such that its complementary function $\Psi(\cdot)$ is continuous, and if G is an additive (scalar) set function of Φ-bounded variation relative to μ, a set function with* FSP, *then G is μ-continuous.*

It suffices to show that ρ', or $N_\Psi(\cdot)$ is continuous. For any $A \in \Sigma$ with $0 < k_0 = N_\Psi(\chi_A) < \infty$, it is seen that (in the definition of (1) equality holds in the present case since Ψ is continuous),

$$1 = \int_\Omega \Psi\left(\frac{\chi_A}{k_0}\right) d\mu = \Psi\left(\frac{1}{k_0}\right) \mu(A),$$

so that

$$k_0 = N_\Psi(\chi_A) = \left[\Psi^{-1}\left(\frac{1}{\mu(A)}\right)\right]^{-1} \to 0$$

when $\mu(A) \to 0$, as desired.

COROLLARY 3.3. *If ρ is continuous, in the sense of the Lemma, then every additive set function on (Ω, Σ, μ) of ρ'-bounded variation relative to μ, is μ-continuous.*

For, since ρ satisfies the weak Fatou property, by definition, its second associate norm $\rho'' = (\rho')'$, satisfies the inequality $\gamma\rho(\cdot) \leqslant \rho''(\cdot) \leqslant \rho(\cdot)$ for some fixed $0 < \gamma \leqslant 1$, by ([2], Theorem 1.2, which holds for arbitrary measures). So $\rho''(\cdot)$ is also continuous, and thus ρ' and its associate norm ρ'' satisfy the hypothesis of the lemma, and the result follows.

REMARK. This corollary slightly extends a result of Gretsky's ([3], Lemma 14 on p. 46) where he assumed, instead, that *every* bounded function of L^ρ has an absolutely continuous norm.

The completeness problem for measures can be given as follows.

THEOREM 3.4. *Let ρ be a function norm on (Ω, Σ, μ) such that its associate norm ρ' is continuous in the sense of Lemma* 3.1. *Let $\mathscr{F}$ be a family of (real) additive set functions on Σ of ρ-bounded variation relative to μ which is assumed σ-finite. Suppose the following conditions hold for $\mathscr{F}$:*

1. *every $\nu \in \mathscr{F}$ satisfies $|\nu(A)| \leqslant k\mu(A)$, $A \in \Sigma$, where k depends on ν but not on A, and $0 < k < \infty$,*

2. *there exists a* $\nu_0 \in \mathscr{F}$ *such that* ν_0 *is positive on every set of positive* μ-*measure*,

3. *if* E_1, E_2 *in* Σ *are disjoint and* $\mu(E_i) > 0$, $i = 1, 2$, *then there is a* $\nu \in \mathscr{F}$ *such that* $\nu(E_1) \leqslant 0$ *and* $\nu(E_2) > 0$.

If f *is any measurable function such that* (i) $\int_\Omega |f| \, d|\nu| < \infty$, *for each* $\nu \in \mathscr{F}$, *where* $|\nu| = \nu^+ + \nu^-$ *is the variation measure of* ν, *and* (ii) $\int_\Omega f \, d\nu = 0$ *for all* $\nu \in \mathscr{F}$, *then* $f = 0$, a.e., $[\mu]$.

REMARK. Even if all ν of $\mathscr{F}$ are probability measures, f need not be bounded so that, in view of the counterexample ([11], p. 152), the conclusion of the theorem is stronger. However, the family is now satisfying a stronger set of conditions (1)–(3) above which naturally are not satisfied by the above counterexample of [11].

PROOF. The hypothesis on the norms ρ and ρ' implies, by Lemma 3.1, that the set functions of $\mathscr{F}$ are all μ-continuous so that condition (1) is meaningful. [The latter condition in general does not ensure ρ-boundedness so that the hypothesis is not redundant!] The μ-continuity here implies every $\nu \in \mathscr{F}$ is countably additive on every restriction σ-field $\Sigma(A)$ of Σ to sets $A \in \Sigma_1$, the ring of sets in Σ of finite μ-measure. A standard argument then implies that each ν is countably additive on the σ-field generated by $\{\Sigma(A) : A \in \Sigma_1\}$ which is Σ itself (cf., e.g., [10], Lemma 8 for a more general case). Then condition (1) and the Radon-Nikodým theorem imply that $\mathscr{F}$ is equivalent to the family $\mathscr{G} = \{g_\nu, \nu \in \mathscr{F}\}$ where $|g_\nu| \leqslant k$, a.e., and where g_ν is the (Radon-Nikodým) derivative of ν relative to μ. Moreover, the ρ-boundedness of ν implies $g_\nu \in L^\rho(\mu)$. Hence $\mathscr{G} \subset L^\rho(\mu)$ and conditions (2) and (3) imply that the algebra $\mathscr{C}_0$ determined by $\mathscr{G}$ in $L^\rho(\mu)$ satisfies the hypothesis of Theorem 2.1.

If $g_0 = (d\nu_0/d\mu)$, then $g_0 > 0$, a.e., and by hypothesis on f, $\int_\Omega |fg| \, d\mu < \infty$, $g \in \mathscr{C}_0$ as well as $\int_\Omega |fg| \, d\nu_0 = \int_\Omega |fgg_0| \, d\mu < \infty$. Thus if $L^\rho(\nu_0)$ is the space based on (Ω, Σ, ν_0), then $L^\rho(\mu) \subset L^\rho(\nu_0)$ and since the measures μ and ν_0 are equivalent, $\mathscr{C}_0 \subset L^\infty(\nu_0)$, and by Theorem 2.1 it follows that $L^\rho(\Sigma_0, \mu) = L^\rho(\Sigma, \mu)$ where $\Sigma_0 = \Sigma(\mathscr{C}_0)$ is the σ-field determined by $\mathscr{C}_0$. If $\mathscr{C}^1$ is the algebra generated by 1 and $\mathscr{C}_0$, then $\Sigma(\mathscr{C}_0) = \Sigma(\mathscr{C}^1)$ and $\mathscr{C}^1$ is dense in $L^\infty(\nu_0)$, because, as in the proof of the Theorem 2.1, $\mathscr{C}^1$ and $L^\infty(\nu_0)$ are both isometrically isomorphic to the space of all continuous functions on a compact space. Also note that $f \in L^1(\nu_0)$. If $x_f^*(g) = \int_\Omega fg \, d\nu_0$, $g \in \mathscr{C}^1$, then $x_f^*(\cdot)$ is a continuous linear functional on $L^\infty(\nu_0)$. Since the class of null sets in Σ for both μ and ν_0 is the same, by hypothesis, it follows that $L^\infty(\nu_0) = L^\infty(\mu)$, that $x_f^*(\cdot)$ is also a continuous linear functional on $L^\infty(\mu)$, and that $\mathscr{C}^1$ is dense in $L^\infty(\mu)$. But $x_f^*(g) = \int_\Omega fg \, d\nu_0 = \int_\Omega fgg_0 \, d\mu = x_{fg_0}^*(g)$, for $g \in \mathscr{C}^1$. By hypothesis $x_f^*(1) = 0$ and $x_f^*(g) = 0$ for all $g \in \mathscr{C}_0 \subset L^\infty(\mu)$, considering x_f^* as a continuous functional on $L^\infty(\mu)$. So $x_f^*(\mathscr{C}^1) = 0$. Since $\mathscr{C}^1$ is dense in

($L^\infty(\mu)$ and hence in) $L^\infty(\nu_0)$, it follows that $x_f^*(L^\infty(\nu_0)) = 0$. Thus $x_f^* = 0$ or $f = 0$, a.e. (ν_0) and so also $f = 0$, a.e. (μ). This completes the proof of the theorem.

SOME REMARKS. 1. It should be noted that the proof of Theorem 2.1 uses only the monotonicity and weak Fatou property of ρ, and thus the result holds, more generally, for certain Fréchet function spaces with monotone distance functions. Thus the proof of that theorem also establishes the following result.

THEOREM 3.5. *Let* (Ω, Σ, μ) *be a complete measure space and* L^ρ *be the (sub-) space of scalar measurable functions* f *on* Σ *such that* $\rho(f) < \infty$, *where* ρ *satisfies*: (i) $\rho(f) = \rho(|f|) \geqslant 0$, (ii) $f_1 \leqslant f_2$, *a.e., implies* $\rho(f_1) \leqslant \rho(f_2)$, (iii) $f_n \nearrow f$ *a.e.*, $\rho(f_n) \leqslant k < \infty$ *implies* $\rho(f) < \infty$, (iv) $\rho(f) < \infty$ *implies* $\rho(af) < \infty$ *for any scalar* a, *and* (v) $\rho(f+g) \leqslant \rho(f) + \rho(g)$. *If* $\mathscr{C}_0 \subset L^\rho$ *is an algebra of essentially bounded functions satisfying conditions* (1)–(3) *of Theorem* 2.1, *then the class of all* Σ_0*-measurable* ρ*-bounded functions is precisely* L^ρ, *where* Σ_0 *is the* σ*-field determined by* $\mathscr{C}_0$.

2. Simple alternate conditions can be given under certain specializations. For instance the following statement can be verified as a consequence of the above result.

COROLLARY 3.6. *If* L^ρ *is a space as in Theorem* 3.5, *on a finite measure space with* $\rho(1) < \infty$, *points of* Ω *are measurable, and if* $\mathscr{C}_0$ *is the algebra generated by* 1 *and a bounded measurable function* f *on* Ω *with the property that either* (i) $f(\omega_1) = f(\omega_2)$ *implies* $\omega_1 = \omega_2$, *or* (ii) f *is strictly increasing, then* $\mathscr{C}_0$ *satisfies the conditions* (1)–(3). *In particular, if* $L^\rho(\Omega, \Sigma, \mu) = L^\Phi(\Omega, \Sigma, \mu)$, $\mu(\Omega) < \infty$, *and* $\Phi(2x) \leqslant c\Phi(x)$ for $|x| \geqslant x_0 > 0$, *then the algebra generated by* 1 *and* f *above is dense in* L^Φ.

3. The assumption of σ-finiteness of μ in Theorem 3.4 is not essential, and the result holds true as stated if only μ has FSP. However, in the proof all functions g must be replaced by "quasi-functions," (i.e., g coincides with a measurable function on every μ-finite set) and then the proof goes through with an additional argument as in (e.g., [9] and [10]). Also in the above work all functions are scalar valued and it will be of interest to extend the Theorem 2.1 if the L^ρ space consists of vector valued functions.

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